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MULTIPLE SOLUTIONS FOR NATURAL CONVECTIVE FLOWS IN AN INTERNALLY HEATED, VERTICAL CHANNEL WITH VISCOUS DISSIPATION AND PRESSURE WORK

D. L. TURCOTTE
Department of Geological Sciences, Cornell University, Ithaca, NY 14853, U.S.A.
D. A. SPENCE
Department of Engineering Science, University of Oxford, Oxford, U.K.
and
H. H. BAU*
Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY 14853, U.S.A.

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Abstract—This paper considers the vertical flow of an internally heated Boussinesq fluid in a vertical channel with viscous dissipation and pressure work. In the absence of internal heating and with no applied pressure gradient, two solutions are obtained; the first is the expected solution with no flow. The second adiabatic solution has temperatures less than the wall temperature and a large downward velocity. For moderate values of heat addition two solutions are obtained; an upper branch with hot temperatures and upward velocities and a lower branch with downward velocities and cool temperatures. When the non-dimensional heat addition parameter $A = Hh^2\alpha^2 g^2/\nu^2 c_p^2 \kappa^2$ reaches a critical value just under 1000 the solutions bifurcate and four solutions are obtained. For large values of A the solutions are examined using the method of matched asymptotic expansions. The equation of the inner solution is of the form of the Painleve transcendent. In the limit of very large A an infinite number of solutions are found for the inner problem.

NOMENCLATURE

A, $= Hh^4 D^2/\nu\kappa^2$, non-dimensional heat production;
b, depth of fluid layer;
c_p, specific heat at constant pressure;
C, $= \nu\kappa/gh^3$, non-dimensional parameter;
D, $= \alpha gh/c_p$, dissipation number;
g, acceleration of gravity;
h, half-width of channel;
H, heat generation per unit mass;
k, thermal conductivity;
p, pressure;
Pr, $= \nu/\kappa$, Prandtl number;
t, time;
T, temperature;
u, velocity;
V, non-dimensional vertical velocity;
x, coordinate.

Greek symbols

α, coefficient of thermal expansion;
γ, $= \alpha/\rho_0 c_p \chi$, Gruneisen's parameter;
η, absolute viscosity;
θ, non-dimensional temperature;
κ, thermal diffusivity;
ν, kinematic viscosity;
τ_{ij}, shear stress;
χ, isothermal compressibility.

INTRODUCTION

VERTICAL channel and pipe flows with viscous dissipation have been considered by a number of authors [1–3]. Although these authors include viscous heating they do not include pressure work. This omission is appropriate if the flow is driven by an external pressure gradient but is not appropriate if it is driven by buoyancy forces. The role of both viscous dissipation and pressure work in thermal convection within a horizontal fluid layer heated from below has been considered by [4]. Both effects have been considered for laminar natural convection on a vertical flat surface by [5].

The roles of viscous dissipation and pressure work during natural convection are particularly important in mantle convection [6]. The governing parameter is the dissipation number $D = \alpha gb/c_p$. In this paper we consider flow in a vertical channel. The principle purpose is to better understand natural convective flows with both viscous dissipation and pressure work.

* Present address: Department of Mechanical Engineering, University of Pennsylvania, Philadelphia, PA 19104, U.S.A.

FORMULATION OF THE PROBLEM

The basic equations for conservation of mass, momentum and energy in a compressible fluid are [7]:

$$\frac{1}{\rho}\frac{D\rho}{Dt}+\frac{\partial u_i}{\partial x_i}=0, \qquad (1)$$

$$\rho\frac{Du_i}{Dt}=-\frac{\partial p}{\partial x}+\frac{\partial \tau_{ij}}{\partial x_j}+\rho g_i, \qquad (2)$$

$$\rho c_p\frac{DT}{Dt}-T\alpha\frac{Dp}{Dt}=k\frac{\partial^2 T}{\partial x_i^2}+\rho H+\tau_{ij}\frac{\partial u_i}{\partial x_j}. \qquad (3)$$

We assume that the changes of density are small so that the equation of state is given by

$$\rho=\rho_0[1+\chi(p-p_0)-\alpha(T-T_0)] \qquad (4)$$

where p_0 and T_0 are a reference pressure and temperature.

In the absence of convection, heat addition and heat conduction we can define an adiabatic solution given by:

$$\frac{\mathrm{d}p_a}{\mathrm{d}y}=-\rho_a g, \qquad (5)$$

$$\frac{\mathrm{d}T_a}{\mathrm{d}p_a}=\frac{\alpha T_a}{c_p\rho_a}. \qquad (6)$$

The solution of these equations with (4) is

$$T_a=T_0\exp\left(-\frac{\alpha g y}{c_p}\right), \qquad (7)$$

$$p_a=p_0+\left(\frac{1+\alpha T_0}{\chi}\right)\times[\exp(-\rho_0 g x y)-1]+\frac{\alpha T_0\rho_0}{\left(\rho_0\chi-\dfrac{\alpha}{c_p}\right)}\times\left[\exp\left(-\frac{\alpha g y}{c_p}\right)-\exp(-\rho_0 x g y)\right]. \qquad (8)$$

It is convenient to introduce two non-dimensional parameters

$$D=\frac{\alpha g h}{c_p}, \qquad (9)$$

$$\gamma=\frac{\alpha}{\rho_0 c_p \chi} \qquad (10)$$

where D is the dissipation number and γ Gruneisen's parameter. We will associate the characteristic length h with the half-width of the vertical channel and introduce non-dimensional coordinates

$$\bar{x}_i=\frac{x_i}{h}. \qquad (11)$$

Introducing (9)–(11) into (7) and (8) gives

$$T_a=T_0\exp(-D\bar{y}), \qquad (12)$$

$$\frac{p_a-p_0}{\rho_0 g h}=\frac{\gamma}{D}(1+\alpha T_0)\left[\exp\left(-\frac{D\bar{y}}{\gamma}\right)-1\right]+\frac{\alpha T_0}{D}\left(\frac{\gamma}{1-\gamma}\right)\left[\exp(-D\bar{y})-\exp\left(-\frac{D\bar{y}}{\gamma}\right)\right]. \qquad (13)$$

Since γ is of order 1 for all fluids and solids that behave like fluids due to solid state creep, the adiabatic quantities change significantly over a vertical distance of order $c_p/\gamma g$. With γ of order 1, a sufficient condition for the density change to be small is $D \ll 1$. In this limit the adiabatic state reduces to

$$T_a=T_0, \qquad (14)$$

$$p_a=p_0-\rho_0 g h\bar{y}, \qquad (15)$$

$$\rho_a=\rho_0. \qquad (16)$$

We next introduce a set of non-dimensional variables:

$$u_i=\frac{\kappa}{h}\frac{\bar{u}_i}{D},\quad p=p_a+\frac{\eta\kappa}{h^2 D}\bar{p},\quad \tau_{ij}=\frac{\eta\kappa}{h^2 D}\bar{\tau}_{ij},$$

$$\rho=\rho_a+\rho_0\bar{\rho},\quad T=T_a+\frac{C\theta}{\alpha D} \qquad (17)$$

where

$$C=\frac{\nu\kappa}{g h^3}. \qquad (18)$$

Introducing these non-dimensional variables into (4) we obtain

$$\rho=\rho_a+\rho_0\frac{C}{\gamma}\bar{p}-\rho_0\frac{C}{D}\theta. \qquad (19)$$

We shall obtain solutions for which θ is of order 1, for these solutions to be incompressible we further require $C/D \ll 1$. If $D \ll 1$ and $C/D \ll 1$ the flow is incompressible and (1) reduces to

$$\frac{\partial \bar{u}_i}{\partial \bar{x}_i}=0. \qquad (20)$$

We next turn to the momentum equation. Introducing the non-dimensional variables into (2) gives

$$\frac{\rho}{\rho_0}\frac{\bar{u}_i}{Pr\,D}\frac{\partial \bar{u}_i}{\partial \bar{x}_j}=-\frac{\partial \bar{p}}{\partial \bar{x}_i}+\frac{\partial \bar{\tau}_{ij}}{\partial \bar{x}_j}+\left(-\frac{D}{\gamma}\bar{p}+\theta\right)\delta_{yi} \qquad (21)$$

where $Pr=\eta/\kappa\rho_0$ is the Prandtl number and we have taken g to be in the $-y$ direction. Assuming the fluid to be incompressible except in the buoyancy term (a modified form of the Boussinesq approximation) we have

$$\bar{\tau}_{ij}=\frac{\partial \bar{u}_i}{\partial \bar{x}_j} \qquad (22)$$

and (21) reduces to

$$\frac{\bar{u}_j}{Pr\,D}\frac{\partial \bar{u}_i}{\partial \bar{x}_j}=-\frac{\partial \bar{p}}{\partial \bar{x}_i}+\frac{\partial^2 \bar{u}_i}{\partial \bar{x}_j^2}+\theta\delta_{yi}. \qquad (23)$$

For a very viscous fluid $Pr\,D \gg 1$, the inertia term can be neglected, and (23) reduces to

$$0=-\frac{\partial \bar{p}}{\partial \bar{x}_i}+\frac{\partial^2 \bar{u}_i}{\partial \bar{x}_j^2}+\theta\delta_{yi}. \qquad (24)$$

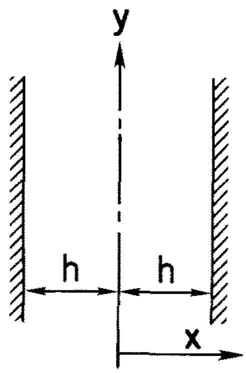

FIG. 1. Illustration of the vertical channel with width $2h$.

Finally we consider the energy equation. Introducing the non-dimensional variables into (3) gives

$$\frac{\rho}{\rho_0}\left[D\bar{u}_i\frac{\partial\theta}{\partial\bar{x}_i} + \theta\bar{u}_y - T_a\alpha\bar{u}_i\frac{\partial\bar{p}}{\partial\bar{x}_i} - \frac{C}{D}\bar{u}_i\frac{\partial\bar{p}}{\partial\bar{x}_i}\right] = \frac{\partial^2\theta}{\partial\bar{x}^2} + \frac{\rho}{\rho_0}A + \bar{\tau}_{ij}\frac{\partial\bar{u}_i}{\partial\bar{x}_j} \quad (25)$$

where

$$A = \frac{Hh^4D^2}{\nu\kappa^2}. \quad (26)$$

We note that $A = RaD$ where $Ra = \alpha g\rho Hh^5/k\kappa\nu$ is the Rayleigh number for a fluid layer of thickness h heated from within. Assuming incompressibility so that $D \ll 1$, $\rho/\rho_0 \simeq 1$, $C/D \ll 1$, $T_a\alpha \ll 1$ and (25) is valid, we obtain

$$\theta\bar{u}_y = \frac{\partial^2\theta}{\partial\bar{x}_i^2} + A + \left(\frac{\partial\bar{u}_i}{\partial\bar{x}_j}\right)^2. \quad (27)$$

Our primary interest in this paper is a vertical, fully developed flow in a channel with a width $2h$ and with walls held at $\theta = 0$ as illustrated in Fig. 1. We assume that (23), (24) and (27) are applicable. For this problem $\bar{u}_i = V(\bar{x})$ and $\theta = \theta(\bar{x})$. We further assume that there is no applied pressure gradient so that $\bar{p} = 0$. We find that (23), (24) and (27) reduce to

$$0 = \frac{d^2V}{d\bar{x}^2} + \theta \quad (28)$$

$$\theta V = \frac{d^2\theta}{d\bar{x}^2} + A + \left(\frac{dV}{d\bar{x}}\right)^2. \quad (29)$$

Eliminating θ from (28) and (29) we obtain

$$\frac{d^4V}{d\bar{x}^4} = \frac{d}{d\bar{x}}\left(V\frac{dV}{d\bar{x}}\right) + A. \quad (30)$$

This is the basic equation that we shall consider in this paper.

The boundary conditions for flow in an isothermal, vertical channel with walls at $\bar{x} = \pm 1$ are

$$V = 0, \quad \theta = 0 \quad \text{at } \bar{x} = \pm 1. \quad (31)$$

Integration of (30) twice with these boundary conditions yields

$$\frac{d^2V}{d\bar{x}^2} = \tfrac{1}{2}V^2 - \tfrac{1}{2}A(1 - \bar{x}^2). \quad (32)$$

The required boundary conditions for this 2nd-order, non-linear differential equation are $V = 0$ at $\bar{x} = \pm 1$.

In our analysis we shall consider only symmetric solutions of (32) with respect to $\bar{x} = 0$. Thus we can replace one of the boundary conditions (31) with

$$\frac{dV}{d\bar{x}} = \frac{d\theta}{d\bar{x}} = 0 \quad \text{at } \bar{x} = 0. \quad (33)$$

We now eliminate θ from (28) and (29) and integrate using (33) with the result

$$\frac{d\theta}{d\bar{x}} = -V\frac{dV}{d\bar{x}} - A\bar{x}. \quad (34)$$

For $\bar{x} = -1$ this becomes

$$\frac{d\theta}{d\bar{x}} = A \geqslant 0. \quad (35)$$

The non-dimensional temperature gradient at the wall is equal to A and is either positive or zero. The heat generated internally in the channel is transferred to the walls.

ADIABATIC SOLUTIONS, $A = 0$

We first consider the solutions when $A = 0$, no heat sources. For this case (32) becomes

$$\frac{d^2V}{d\bar{x}^2} = \tfrac{1}{2}V^2 \quad (36)$$

with the required boundary conditions $V = 0$ at $\bar{x} = \pm 1$. There are two solutions of (36) that satisfy these boundary conditions. The first is $V = 0$, i.e. there is no flow.

A second solution is obtained by integrating (36) with respect to V with $dV/dx = 0$ and $V = V_0$ at $\bar{x} = 0$ to give

$$\left(\frac{dV}{d\bar{x}}\right)^2 = \tfrac{1}{3}(V^3 - V_0^3). \quad (37)$$

Clearly $V_0 < 0$ since $(dV/d\bar{x})^2$ is positive definite and $0 \geqslant V \geqslant V_0$ which implies a negative or downward flow at the center.

We introduce $w = V/V_0$, integrating (37) then gives

$$\int_0^w \frac{dw}{(1 - w^3)^{1/2}} = \left(-\frac{V_0}{3}\right)^{1/2}(1 + \bar{x}) \quad (38)$$

where w is an elliptic function [8]. Since $w = 1$ at $\bar{x} = 0$ the value of V_0 is given by

$$V_0 = -3\left[\int_0^1 \frac{dw}{(1 - w^3)^{1/2}}\right]^2 = -\frac{1}{3}\left[\frac{\Gamma(\frac{1}{3})\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{6})}\right]^2 = 5.89835. \quad (39)$$

Substituting dimensional quantities the maximum velocity at the center of the channel is

$$V(0) = -5.89835\left(\frac{k}{\rho g\alpha h^2}\right). \quad (40)$$

The solution $V = 0$ is expected in the absence of internal heating. The second solution requires further discussion. Cold fluid is introduced at the top of the channel and flows down the channel at constant velocity without any applied pressure gradient. The maximum downward velocity is given by (37). Since $A = 0$, $d\theta/d\bar{x} = 0$ at $\bar{x} = \pm 1$ from (35), thus there is no heat flux at the walls, the second solution is adiabatic.

From (28) and (36) we find that

$$\theta = \tfrac{1}{2}V^2. \quad (41)$$

From (39) and (41) the minimum temperature at the center of the channel is

$$\theta(0) = -\tfrac{1}{2}V_0^2 = -17.39527 \quad (42)$$

and the actual temperature difference between the center of the channel and the walls is

$$T - T_0 = -17.39527\frac{gc_p h^2}{d^2 g\nu\kappa}. \quad (43)$$

We shall refer to this adiabatic solution as the lower branch solution and the trivial $\theta = 0$ solution as the upper branch solution. We shall show that for small and intermediate values of A two solutions are obtained which can be associated with these two branches.

SOLUTION FOR SMALL A

We next consider a solution for the upper branch valid for small A. Differentiation of (32) with respect to A gives

$$\frac{\partial^2}{\partial\bar{x}^2}\left(\frac{\partial V}{\partial A}\right) = V\frac{\partial V}{\partial A} - \tfrac{1}{2}(1 - \bar{x}^2). \quad (44)$$

Our objective is to obtain the value of $\partial V/\partial A$ at $A = 0$. Since $V = 0$ at $A = 0$ for the upper branch solution, (44) can be integrated twice to give

$$\left.\frac{\partial V}{\partial A}\right|_{A=0} = \tfrac{1}{24}(1 - \bar{x}^2)(5 - \bar{x}^2). \quad (45)$$

The constants of integration have been evaluated using the conditions $dV/dA = 0$ at $\bar{x} = \pm 1$. Therefore the upper branch solution for V is initially an increasing function of A since $dV/dA > 0$ for $-1 < \bar{x} < 1$.

An iteration procedure valid for small A is developed and is used to prove the existence and uniqueness of the upper branch solution for a certain range of A. Integrating (32) twice with respect to $\bar{x}$ and applying the boundary conditions $V(-1) = dV/d\bar{x}(0) = 0$ gives the non-linear integral equation

$$V(\bar{x}) = -\frac{1}{2}\int_{-1}^{0} k(\bar{x}, t)\,V^2(t)\mathrm{d}t + Ag(\bar{x}),$$

$$k(x, t) = \begin{cases} 1 + \bar{x} & t \le \bar{x} \le 0 \\ 1 + t & -1 \le t \le \bar{x} \end{cases}, \quad (46)$$

$$g(\bar{x}) = \tfrac{1}{24}(1 - \bar{x}^2)(5 - \bar{x}^2).$$

This is an equation of the Hammerstein type. It can be shown that for a certain range of A the iterative procedure

$$V_{n+1}(\bar{x}) = -\frac{1}{2}\int_{-1}^{0} k(\bar{x}, t)\,V_n^2(t)\mathrm{d}t + Ag(\bar{x}) \quad (47)$$

with

$$V_0(\bar{x}) = Ag(\bar{x}) \quad (48)$$

converges uniquely to a function $V(\bar{x})$ satisfying (46).

Since k is positive it follows from (47) that $V_{n+1} \le Ag$ for all x. Then from (49) we have

$$|V_{n+1} - V_n| = +\frac{1}{2}\int_{-1}^{0} k\,|V_n^2 - V_{n-1}^2|\,\mathrm{d}\bar{x}$$

$$\le \frac{A}{2}\int_{-1}^{0} k\,|V_n - V_{n-1}|\cdot|2g|\,\mathrm{d}\bar{x} \quad (49)$$

and the L_2 norm, defined for a function f by

$$\|f\| = \left(\int_{-1}^{0} |f|^2\,\mathrm{d}\bar{x}\right)^{1/2} \quad (50)$$

satisfies the inequality

$$\|V_{n+1} - V_n\| \le AM\,\|V_n - V_{n-1}\| \quad (51)$$

where

$$M^2 = \int_{-1}^{0}\int_{-1}^{0} k^2(\bar{x}, t)\,g^2(t)\mathrm{d}\bar{x}\,\mathrm{d}t \quad (52)$$

giving

$$M = 0.0736628. \quad (53)$$

Convergence of the iterative procedure is guaranteed for $A < M^{-1} = 13.575$.

A numerical solution based on (49) led to con-

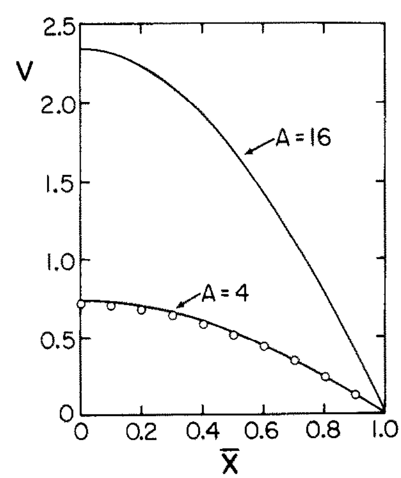

FIG. 2. Velocity profiles of the upper branch solution for A = 4 and 16. Solid lines are iterative solutions of (49). Circles are from the series solution given in (56).

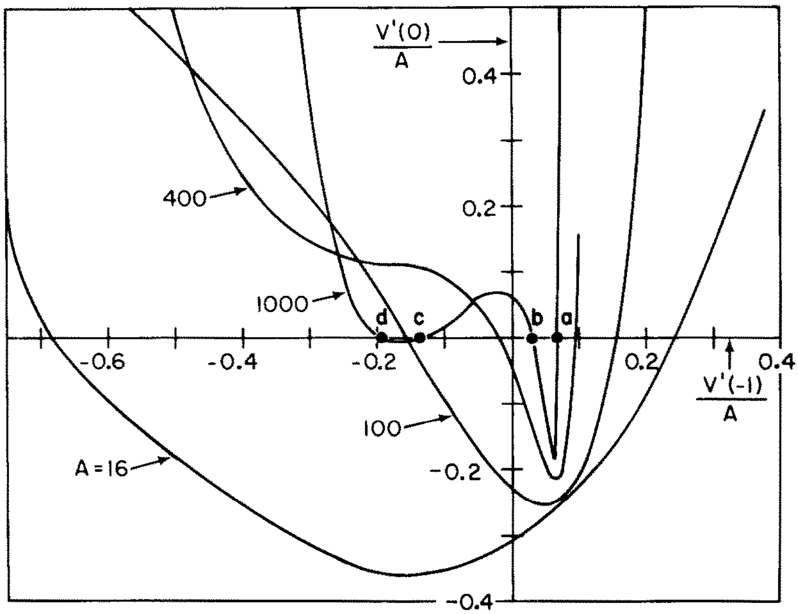

FIG. 3. Dependence of $(dV/d\bar{x})A^{-1}$ at $x = 0$ on $(dV/d\bar{x})A^{-1}$ at $x = -1$. Since solutions must satisfy the condition $(dV/dx) = 0$, valid solutions correspond to the intersections of the curves with the horizontal axis. For $A = 16$, 100 and 400 two solutions are obtained. For $A = 1000$ four solutions are obtained, these are denoted as a, b, c and d.

vergence for $A \le 20$ and failed to converge for larger values of A. The numerical solution utilized a 10 point Gaussian integration formula. The iteration was considered to converge if

$$\left| \sum_{i=1}^{10} V_{n+1}^2(\bar{x}_i) - \sum_{i=1}^{10} V_n^2(\bar{x}_i) \right| < \varepsilon \qquad (54)$$

where x_i were the points where the function was evaluated. The number of iterations required for convergence increased with increasing A. Velocity profiles for $A = 4$ and 16 with $\varepsilon = 10^{-3}$ are given in Fig. 2.

An alternative technique for obtaining solutions valid for small A is to expand in an asymptotic series of the form

$$V = AV_0 + A^2 V_1 + \cdots. \qquad (55)$$

Substitution into (32) gives

$$V = \frac{1}{24}(\bar{x}^4 - 6\bar{x}^2 + 5)A + \frac{1}{384}\left(\frac{x^{10}}{270} - \frac{x^8}{14} + \frac{23x^6}{45} - \frac{5x^4}{3} + \frac{25x^2}{6} - 2.94\right)A^2 + O(A^3). \qquad (56)$$

Values obtained from this two term expansion are compared with the iterative solution for $A = 4$ in Fig. 2. Good agreement is found.

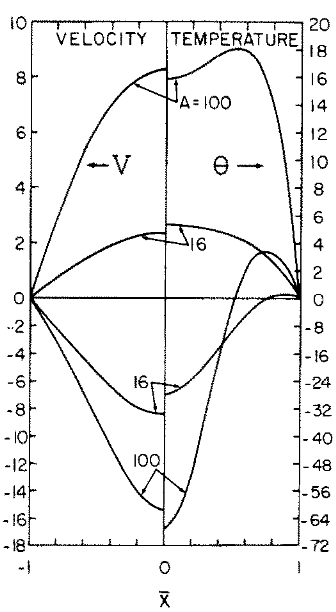

FIG. 4. Velocity and temperature profiles for $A = 16$ and 100 obtained by numerical integration of (32).

SOLUTIONS FOR INTERMEDIATE VALUES OF A

Numerical solutions for the distribution of velocity and temperature in the channel have been obtained for intermediate values of A. We have solved (32) using the Runge–Kutta integration technique. We look for symmetric solutions in $\bar{x}$ so require solutions for which $dV/d\bar{x} = 0$ at $\bar{x} = 0$. In addition we require $V = 0$ at $\bar{x} = -1$. Since our solutions are symmetric we consider only the region $-1 < \bar{x} < 0$. In order to solve this two point boundary value problem we apply the boundary condition $V(-1) = 0$ and choose an arbitrary value $V'(-1)$, ($V' \equiv dV/d\bar{x}$). A numerical integration of (32) is then carried out for $-1 < \bar{x} < 0$. In general the integration will yield $V'(0) \neq 0$. The dependence of

$V'(0)/A$ on $V'(-1)/A$ is given in Fig. 3 for $A = 16, 100, 400$ and 1000.

Noting that a solution to our problem requires $V'(0)=0$ it is clear that we have two solutions for $A=16$ corresponding to $V'(-1)/A = 0.245$ and -0.678. The corresponding velocity and temperature profiles are given in Fig. 4. The numerical solution for the upper branch solution is in excellent agreement with the iterative solution given in Fig. 2. The velocities associated with the lower branch solution are always negative but the temperatures are positive near wall with a negative core. Heat is lost to the wall as predicted by (35).

For $A = 100$ there are also two solutions; from Fig. 3 these correspond to $V'(-1)/A = 0.155$ and -0.150. The corresponding velocity and temperature profiles for the two solutions are given in Fig. 4. For the upper solution the velocities and temperatures are both positive; the maximum velocity is at the center of the channel but there are two temperature maximums near $\bar{x} = \pm 0.5$. For the lower solution the velocity is negative everywhere but the temperature is negative only in the center of the channel.

From Fig. 3 we see that there are also two solutions for $A = 400$. However, for $A = 1000$ there are four solutions. A bifurcation occurs allowing two additional solutions. The velocity profiles corresponding to the four solutions are given in Fig. 5. The upper solution continues to have positive velocities, however, the lower solution has a region of positive velocities near the walls and a negative velocity core. The two new solutions have maximum negative velocities near $\bar{x} = \pm 0.6$. One of these solutions has a small region of positive velocity near the center of the channel.

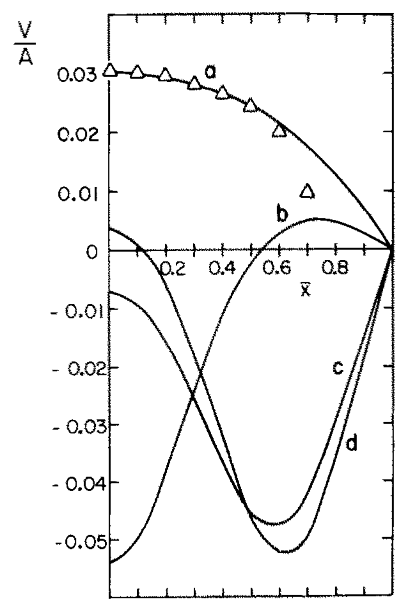

FIG. 5. Velocity profiles for $A = 1000$. The four solutions a, b, c and d are obtained by a numerical integration of (32).

SOLUTION FOR LARGE A

In order to gain further insight into the solutions valid for large A we utilize the method of matched asymptotic expansions. In order to obtain the outer solution valid in the core flow we rewrite (32) as

$$V^2 = A(1-\bar{x}^2) + 2\frac{d^2V}{d\bar{x}^2}. \tag{57}$$

We expand V in powers of $A^{-1/2}$. The leading term V_0 is obtained by neglecting $d^2V/d\bar{x}^2$ in (57) with the result

$$V_0 = \pm[A(1-\bar{x}^2)]^{1/2}. \tag{58}$$

To obtain the next term in the expansion we write

$$V^2 = A(1-\bar{x}^2) + 2\frac{d^2V_0}{d\bar{x}^2}$$

$$= A(1-\bar{x}^2) \pm 2\frac{d^2}{d\bar{x}^2}[A(1-x^2)]^{1/2} \tag{59}$$

and obtain

$$V = \pm[A(1-\bar{x}^2)]^{1/2} \mp (1-\bar{x}^2)^{-2} + O(A^{-1/2}). \tag{60}$$

The upper and lower signs correspond to the upper and lower branches respectively. Only the first term of the expansion satisfies the boundary conditions $V=0$ at $x = \pm 1$; and this term does not satisfy the differential equation. In order to complete the solution an inner solution is required that is valid near the walls. The outer solution for the upper branch solution is compared with the solution "a" for $A = 1000$ in Fig. 5. We will next consider the inner solution valid near $x = -1$.

The appropriate inner variables are

$$\bar{x} = -1 + \varepsilon X, \quad V = \varepsilon^{-2} Y \tag{61}$$

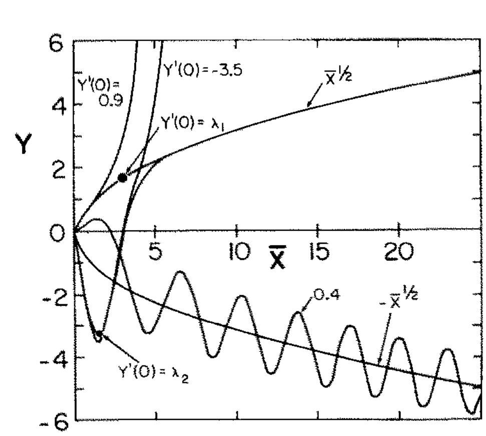

FIG. 6. Velocity solutions of (63) in terms of the inner variables valid near the walls. Solutions corresponding to the limiting values $Y'(0) = \lambda_1$ and λ_2 are given. These approach the outer solution $X^{1/2}$. A typical intermediate solution $Y'(0) = 0.4$ approaches the outer solution $-X^{1/2}$ in an oscillatory manner. Two divergent solutions, $Y'(0) = 0.9, -3.5$ are given.

where

$$\varepsilon = (2A)^{-1/5}. \qquad (62)$$

Substitution into equation (57) gives

$$2\frac{d^2 Y}{dX^2} = Y^2 - X. \qquad (63)$$

The solution must satisfy the boundary condition $Y = 0$ at $X = 0$ and Y must match the inner expansion of the outer solution as $\bar{X} \to \infty$. This limit, written in terms of the inner variables, is $Y \to \pm X^{1/2}$ as $X \to \infty$.

The equation satisfied by the inner variables, (63), is of the form of the Painlevé transcendent. Its behavior near the origin has been studied by Holmes and Spence [9]. It was found that there are two solutions $Y(X)$ passing through the origin that approach $+X^{1/2}$ as $X \to \infty$. The values of their slopes at the origin are

$$\bar{Y}'(0) = \lambda_1 = 0.804715499 \ldots,$$

$$\bar{Y}'(0) = \lambda_2 = -3.3010879 \ldots.$$

Some numerical solutions of (63) are given in Fig. 6. All solutions pass through the origin and have different values for the initial slope. Three types of behavior are observed: (i) For $Y'(0) > \lambda_1$ and $Y'(0) < \lambda_2$ the function $Y(X)$ grows exponentially with X and approaches $+\infty$. Examples are given for $Y'(0) = 0.9$ and -3.5. These solutions have double poles at some positive X; (ii) for $\lambda_1 > Y'(0) > \lambda_2$ the solutions are of the form of slowly decaying oscillations about $-X^{1/2}$. These solutions approach $-X^{1/2}$ as $X \to \infty$. This oscillatory behavior was previously observed for $A = 1000$ and was illustrated in Fig. 5. The solution for $Y'(0) = 0.4$ is given in Fig. 6; (iii) for $Y'(0) = \lambda_1$ and $Y'(0) = \lambda_2$ the solutions approach $+X^{1/2}$ as $X \to \infty$ as illustrated in Fig. 6.

The inner expansion of the outer solution (60) in terms of the inner variables gives two solutions $Y = \pm X^{1/2}$. Thus two solutions corresponding to $\bar{Y}'(0) = \lambda_1, \lambda_2$ approach $Y = +X^{1/2}$ asymptotically and an infinite number of solutions corresponding to $\lambda_2 < Y'(0) < \lambda_1$ approach $Y = -X^{1/2}$ asymptotically but much more slowly because of their oscillatory behavior.

For finite but large values of A those solutions that satisfy the symmetry condition $V'(0) = 0$ are allowed solutions. For $A = 1000$ we found four such solutions numerically. The solution denoted "a" (see Fig. 5) had the largest value of $V'(-1)$, i.e. the largest value of λ in terms of inner variables, and therefore appears to correspond to the solution for $\lambda = \lambda_1$. The solution denoted "d" (see Fig. 5) had the smallest value of $V'(-1)$, i.e. the smallest value of λ in terms of inner variables, and therefore appears to correspond to the solution for $\lambda = \lambda_2$. The solutions "b" and "c" are members of the oscillatory family of solutions with $\lambda_1 < \lambda < \lambda_2$. Presumably for larger values of A more and more solutions will be obtained that satisfy the necessary boundary conditions.

APPLICATION TO THE EARTH'S MANTLE

It is now generally recognized that thermal convection in the earth's mantle is responsible for plate tectonics and mantle convection. The crystalline mantle behaves as a fluid on geological time scales due to solid state creep processes. The mantle is heated from within due to the decay of the radioactive isotopes of uranium, thorium and potassium.

It is of interest to determine whether the processes considered in this paper are relevant to mantle convection. In order to do this we consider relevant parameter and scale values for the earth's mantle. Representative values are (Oxburgh and Turcotte, 1978): $h = 100$ km, $g = 10\,\mathrm{m/s^2}$, $\kappa = 1\,\mathrm{mm^2/s}$, $\nu = 10^{16}\,\mathrm{m^2/s}$, $\alpha = 10^{-5}\,\mathrm{K^{-1}}$, $H = 10^{-11}$ W/kg, $c_p = 1$ kJ/kg K. For the relevant non-dimensional parameters we find that $D = 10^{-2}$, $C = 10^{-6}$ and $A = 10$; thus $C/D = 10^{-4}$ and $D \ll 1$ and $C/D \ll 1$ so the conditions for the validity of the theory are satisfied.

We are particularly interested in the properties of the lower branch solution for $A \simeq 10$. The maximum non-dimensional downward velocity is $V \simeq 7$ and the corresponding dimensional velocity is $v = 200$ mm/yr. This is about twice the average velocity of 100 mm/yr associated with plate tectonics. The maximum non-dimensional negative temperature is $\theta \simeq -10$ and the corresponding temperature difference is 180°C. This temperature difference is easily associated with mantle flows. Although the implications of the above results are not clear, it is of interest that the parameter values associated with mantle convection are consistent with the lower branch solution.

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REFERENCES

1. S. Ostrach, Laminar natural convection flow and heat transfer of fluids with and without heat sources in channels with constant wall temperature, NACA TN 2863 (1952).
2. T. M. Hallman, Combined forced and free-laminar heat transfer in vertical tubes with uniform internal heat generation, *Trans. ASME* **78,** 1831–1841 (1956).
3. M. Iqbal, B. D. Aggarwala and M. S. Rokerya, Viscous dissipation effects on combined free and forced convection through vertical circular tubes, *J. Appl. Mech.* **92,** 931–935 (1970).
4. D. L. Turcotte, A. T. Hsui, K. E. Torrance and G. Schubert, Influence of viscous dissipation on Benard convection, *J. Fluid Mech.* **64,** 369–374 (1974).
5. J. A. D. Ackroyd, Stress work effects in laminar flat-plate natural convection, *J. Fluid Mech.* **62,** 677–696 (1974).
6. E. R. Oxburgh and D. L. Turcotte, Mechanisms of continental drift, *Rep. Prog. Phys.* **41,** 1249–1312 (1978).
7. G. K. Batchelor, *An Introduction to Fluid Dynamics.* Cambridge University Press, Cambridge (1967).
8. M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions.* Dover, New York (1965).
9. P. Holmes and D. A. Spence, in preparation, 1982.

SOLUTIONS MULTIPLES POUR DES ECOULEMENTS DE CONVECTION NATURELLE DANS UN CANAL CHAUFFE INTERIEUREMENT, VERTICAL AVEC DISSIPATION VISQUEUSE ET TRAVAIL DE PRESSION

Résumé—On considère l'écoulement vertical d'un fluide de Boussinesq dans un canal vertical, avec dissipation visqueuse et travail de pression. En l'absence d'un chauffage interne et sans application d'un gradient de pression, deux solutions sont obtenues; la première est la solution attendue de l'absence d'écoulement. La seconde solution donne des températures inférieures à celle de la paroi et avec une grande vitesse descendante. Pour des valeurs modérées de l'addition de chaleur, deux solutions sont obtenues; une branche supérieure avec des températures élevées et des vitesses ascendantes et une branche inférieure avec des vitesses ascendantes et des températures basses. Lorsque le paramètre adimensionnel d'addition de chaleur $A = Hh^2\alpha^2g^2/v^2c_p^2\kappa^2$ atteint une valeur critique juste inférieure à 1000 les solutions bifurquent et on obtient quatre solutions. Pour de grandes valeurs de A, les solutions sont examinées en utilisant la méthode des développements asymptotiques. L'équation de la solution interne est de la forme de la transcendante de Painlevé. Dans la limite des très grandes valeurs de A, on trouve pour le problème interne un nombre infini de solutions.

MEHRFACHE LÖSUNGEN FÜR STRÖMUNGEN IN EINEM VERTIKALEN KANAL BEI FREIER KONVEKTION UNTER BERÜCKSICHTIGUNG VON INNERER ERWÄMUNG, VISKOSER DISSIPATION UND DRUCKARBEIT

Zusammenfassung—Diese Arbeit behandelt die vertikale Strömung eines Boussinesq-Fluids mit innerer Erwärmung in einem vertikalen Kanal bei viskoser Dissipation und Druckarbeit. Bei Fehlen von innerer Erwärmung und ohne Druckgradienten werden zwei Lösungen erhalten. Die erste ist die erwartete Lösung ohne Strömung. Die zweite adiabate Lösung zeigt Temperaturen unterhalb der Wandtemperatur und eine große abwärtsgerichtete Geschwindigkeit. Für mäßige Werte der Wärmezufuhr werden zwei Lösungen erhalten: ein oberer Zweig mit hohen Temperaturen und aufwärts gerichteter Strömung und ein unterer Zweig mit abwärts gerichteter Strömung und niedrigen Temperaturen. Wenn der dimensionslose Parameter für die Wärmezufuhr $A = Hh^2\alpha^2g^2/v^2c_p^2\kappa^2$ einen kritischen Wert knapp unter 1000 erreicht, teilen sich die Lösungen, und man erhält vier Lösungen. Für große Werte von A werden die Lösungen mit Hilfe der Methode angepaßter asymptotischer Entwicklung geprüft. Die Gleichung für die innere Lösung ist von der Form der Painleve-Transzendenten. Im Grenzfall sehr großer Werte von A findet man eine unendliche Zahl von Lösungen für das innere Problem.

ВЕТВЛЕНИЕ РЕШЕНИЙ ДЛЯ ЕСТЕСТВЕННОКОНВЕКТИВНЫХ ТЕЧЕНИЙ В НАГРЕВАЕМОМ ИЗНУТРИ ВЕРТИКАЛЬНОМ КАНАЛЕ С УЧЕТОМ ВЯЗКОЙ ДИССИПАЦИИ И РАБОТЫ СИЛ ДАВЛЕНИЯ

Аннотация — Рассматривается течение жидкости Буссинеска, нагреваемой внутренними источниками тепла, в вертикальном канале с учетом влияния вязкой диссипации и работы сил давления. Для случая отсутствия внутреннего источника тепла и градиента давления получены два решения; первое является известным решением для неподвижной жидкости. Во втором адиабатическом решении получены значения температуры жидкости, меньшие значений температуры стенки, и бóльшая скорость вниз по потоку. Для средних значений подвода тепла получены два решения: верхняя ветвь имеет бóльшие значения температуры и направленные вверх скорости, а нижняя ветвь — направленные вниз скорости и малые значения температуры. Когда безразмерный параметр подвода тепла $A = Hh^2\alpha^2g^2/v^2cp^2\kappa^2$ достигает критического значения, равного примерно 1000, два решения распадаются на четыре. При больших значениях A решения находятся методом сращиваемых асимптотических разложений. Внутреннее решение имеет вид трансцендентного уравнения Пейнлее. В пределе очень больших значений A получено бесконечное число решений для внутренней задачи.